Snow College Mathematics Contest

key

April 2, 2019 **Senior Division: Grades 10-12** **Form: T**

Bubble in the single best choice for each question you choose to answer.

1. Find $\log_{10}(\log_{10}(\log_{10} 10^{10\,000\,000\,000}))$.

   (A) 0
   $\boxed{\text{(B)}}$ 1
   (C) 2
   (D) 3
   (E) 6

   SOLN $\log_{10}\left(\log_{10}\left(\log_{10}\left(10^{10^{10}}\right)\right)\right) = \log_{10}\left(\log_{10} 10^{10}\right) = \log_{10}(10) = 1$ □

2. In astronomy, the apparent brightness b of a star is related to the luminosity L by
$$b = \frac{L}{4\pi d^2}$$
where d is the distance to the star. If planet 1 is five times farther from a star than planet 2 is, what will be the ratio b_1/b_2?

   $\boxed{\text{(A)}}$ $\frac{1}{25}$
   (B) $\frac{1}{5}$
   (C) 1
   (D) 5
   (E) 25

   SOLN This is called an inverse-square law. d is in the denominator and it is squared, so double the distance means one quarter the brightness. $\left(\frac{1}{5^2}\right) = \frac{1}{25}$ □

3. A semiprime is the product of two (not necessarily distinct) primes. They are very useful in cryptology because it is easy to multiply two primes together, but hard to factor a large semiprime. What is the sum of the semiprimes less than 20?

   (A) 45
   (B) 49
   (C) 54
   (D) 57
   $\boxed{\text{(E)}}$ 58

   SOLN $4 + 6 + 9 + 10 + 14 + 15 = 58$ □

4. Simplify:
$$\frac{\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2 \sqrt[4]{32\sqrt[3]{8}}}$$

   (A) $\frac{5}{6}$
   (B) $\frac{3^{43/12}}{2^{3/2}}$
   (C) $\frac{\log_3 2}{\log_2 3}$
   (D) $\sqrt[3]{3}$
   $\boxed{\text{(E)}}$ $\frac{43}{18}$

   SOLN Numerator:
$$\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}$$
$$= \log_3\left(3^5\left(3^4 \cdot 3^{1/3}\right)^{1/2}\right)^{1/2}$$
$$= \log_3 3^{43/12} = 43/12$$

   Denominator:
$$\log_2 \sqrt[4]{32\sqrt[3]{8}} = \log_2 64^{1/4} = \log_2 2^{3/2} = \frac{3}{2}$$

   Combine: $\dfrac{43/12}{3/2} = 43/18$ □

5. Let n be a positive integer; then n^2 is a perfect square. How many values of n exist such that $n^2 + 45$ is itself a perfect square?

(A) 0

(B) 1

(C) 2

$\boxed{\text{(D)}}$ 3 $\qquad n = 2,\ 6,\ 22$

(E) 4

$\boxed{\mathcal{SOLN}}$ Say $n^2+45 = m^2$ (a perfect square).

$$45 = m^2 - n^2$$
$$45 = (m+n)(m-n)$$
$$45 = 45 \times 1 = 15 \times 3 = 9 \times 5$$

$$m+n = 45, \quad m-n = 1 \implies$$
$$m = 23, \quad n = 22$$
$$m+n = 15, \quad m-n = 3 \implies$$
$$m = 9, \quad n = 6$$
$$m+n = 9, \quad m-n = 5 \implies$$
$$m = 7, \quad n = 2$$

□

6. Circle with center O has radius 3, $AB = 8$, and $\overline{AB}$ is tangent to the circle at B. If $\overline{BC}$ is a diameter of the circle, find CD.

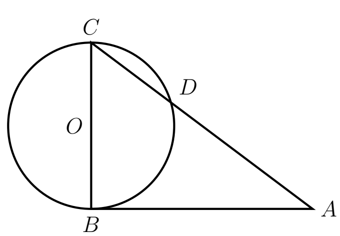
(A) 3

(B) 3.2

$\boxed{\text{(C)}}$ 3.6

(D) 4

(E) 6.4



$\boxed{\mathcal{SOLN}}$ ΔABC is a right triangle with side lengths 8, 6, and 10 respectively. $\Delta BDC \sim \Delta ABC$ so $\frac{CD}{6} = \frac{6}{10}$. □

7. What is the probability that the product of the numbers rolled on three fair six-sided dice is prime?

(A) $\frac{1}{36}$

$\boxed{\text{(B)}}$ $\frac{1}{24}$

(C) $\frac{1}{16}$

(D) $\frac{1}{12}$

(E) $\frac{1}{8}$

$\boxed{\mathcal{SOLN}}$ One die must be prime and the others 1. There are 3 choices for the prime (2, 3, or 5), and 3 choices for which die is prime, so 9 ways this can happen. There are 6^3 outcomes possible, so the probability is $9/6^3$. □

8. A triangle with integer side lengths and positive area has no two sides equal. Find its least possible perimeter.

(A) 6

(B) 7

(C) 8

$\boxed{\text{(D)}}$ 9

(E) 10

$\boxed{\mathcal{SOLN}}$ Each side must be less than the sum of the other two; thus $1+2+3$ is not possible. The next possibility that works: $2+3+4=9$. □

9. A positive integer x less than 35 satisfies the two congruences simultaneously:

$$\begin{aligned} x &\equiv 2 \mod 5 \\ x &\equiv 3 \mod 7 \end{aligned}$$

Which represents $2x \mod 35$?

$\boxed{\text{(A)}}$ -1

(B) 0

(C) 1

(D) 2

(E) 3

$\boxed{\mathcal{SOLN}}$ The modulo function yields the remainder after division: $x \equiv y \mod z$ if $x \div z$ gives a remainder of y. By trial and error and/or checking all solutions of one of the equations, we pinpoint $x = 17$. Then, $2x = 34 \equiv -1 \mod 35$. □

10. A die is *fair* if all faces are congruent and each number has an equal chance of being rolled. Which is not a possible number of faces for a fair die?

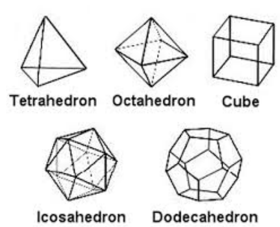
(A) 4

$\boxed{\text{(B)}}$ 5

(C) 8

(D) 12

(E) 20



`https://tinyurl.com/y49yu9q2`

$\boxed{\mathcal{SOLN}}$ All of the Platonic solids are symmetrical isohedra (same faces), as players of role-playing games know. A fair die can't be made with an odd number of faces. Under a different definition of fair (not requiring congruent faces), there can be a clever trick to create a "fair" die which comes up with numbers 1–5 with equal probability. `https://en.wikipedia.org/wiki/Dice` □

11. The Dirac delta function is defined by

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\ \mathrm{d}x = 1$$

Which of the following functions would the Dirac delta function be a derivative of?

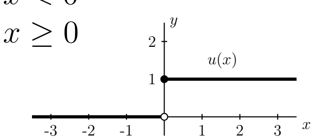
(A) $p(x) = x!$

$\boxed{\text{(B)}}$ $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

(C) $g(x) = |x|$

(D) $r(x) = \frac{1}{1-x}$

(E) $s(x) = \frac{x-1}{1-x}$



$\boxed{\mathcal{SOLN}}$ $u(x)$ is called the unit step function; its derivative is zero everywhere except at the origin. The second part of the definition is not needed to answer this question. □

12. In 1911, physicist Ernest Rutherford discovered that alpha particles shot toward the nucleus of an atom are repulsed away along hyperbolic paths. If a particle gets as close as 5 units to the nucleus with a slant asymptote of $y = \frac{2}{3}x$, what is the mathematical model representing the path?

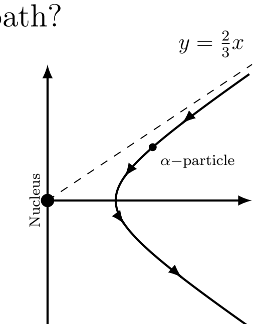
$\boxed{\text{(A)}}$ $\frac{x^2}{25} - \frac{9y^2}{100} = 1$

(B) $\frac{x^2}{5} - \frac{3y^2}{10} = 1$

(C) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

(D) $\frac{x^2}{25} - \frac{y^2}{56.25} = 1$

(E) $\frac{x^2}{3} - \frac{y^2}{2} = 1$



$\boxed{\mathcal{SOLN}}$ $a = 5$ is the distance from center to vertex. The slope of the asymptote of a horizontal hyperbola is $\frac{b}{a} = \frac{b}{5} = \frac{2}{3}$. This implies $b = \frac{10}{3}$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies \frac{x^2}{5^2} - \frac{y^2}{(10/3)^2} = 1$$ □

13. How many different ways can the letters in the word `STEMS` be arranged?

    (A) 16

    (B) 25

    (C) 45

    $\boxed{\text{(D)}}$ 60

    (E) 120

    $\boxed{\text{SOLN}}$ There are five letters, but one repeats. $\frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60$ $\square$

14. Three boys entered a deli. The first ordered 4 sandwiches, a drink, and 10 donuts for \$8.45. The second ordered 3 sandwiches, a drink, and 7 donuts for \$6.30. How much did the third boy pay for a sandwich, a drink, and a donut?

    $\boxed{\text{(A)}}$ \$2.00

    (B) \$2.05

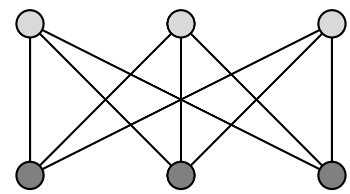
    (C) \$2.10

    (D) \$2.15

    (E) \$2.20

    $\boxed{\text{SOLN}}$ From the first boy we find the price of 8 sandwiches, 2 drinks, and 20 donuts = \$16.90. From the payment of the second we calculate the price of 9 sandwiches, 3 drinks, and 21 donuts = \$18.90. The difference of the sums \$18.90 − \$16.90 = \$2.00 is exactly the price of one sandwich, a drink, and a donut. $\square$

15. In *graph theory*, a graph is composed of vertices (dots) and edges (lines). What is the minimum number of vertex colors required so that no two connected vertices in the graph shown share the same color?

    (A) 1

    $\boxed{\text{(B)}}$ 2

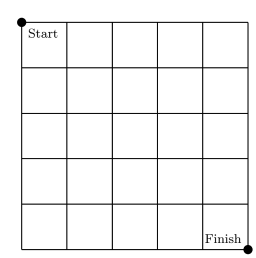
    (C) 3

    (D) 4

    (E) 5

    $\boxed{\text{SOLN}}$ This graph is called a bipartite graph; it can be characterized as a graph whose minimum vertex coloring is two because the graph can be decomposed into two sets of vertices X and Y where vertices in X are not adjacent (connected) to each other and vertices in Y are not adjacent to each other. $\square$

16. Below is a 5×5 grid. A path from "Start" to "Finish" is a sequence of moves right or down along the grid lines in the figure. Left and up are not allowed. How many paths are there from "Start" to "Finish"?

    (A) $\binom{5}{2} \cdot \binom{5}{2}$

    (B) $\binom{10}{2} \cdot 5!$

    (C) $\binom{10}{5} \cdot \binom{10}{5}$

    (D) $5! \cdot 5!$

    $\boxed{\text{(E)}}$ $\binom{10}{5}$

    $\boxed{\text{SOLN}}$ To move from "Start" to "Finish", one must at some point move right 5 and down 5 giving a sequence of 10 moves. If x is a move right or down, then a path looks like a sequence of moves $(x, x, x, x, x, x, x, x, x, x)$. We need to choose 5 of these 10 moves to be right. The remaining moves must be down. Thus, we have $\binom{10}{5}$ possible paths. $\square$

17. Let n be a nonnegative integer. The gamma function, $\Gamma(n)$, is a generalization of the factorial function $n!$. A recursive definition for the gamma function is $\Gamma(n+1) = n\Gamma(n)$, where $\Gamma(1) = 1$. Compute $\Gamma(5)$.

(A) 6

(B) 10

(C) 15

(D) 24

(E) 120

SOLN Computing, we have

$$\begin{aligned}
&\Gamma(1) = 1 \\
&\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1 \cdot 1 \\
&\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 \cdot 1 \\
&\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3 \cdot 2 \cdot 1 \\
&\Gamma(5) = \Gamma(4+1) = 4\Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1
\end{aligned}$$

$\Gamma(z) = \int_0^\infty t^{z-1}\, \mathrm{e}^{-t}\, \mathrm{d}t$ □

18. Ted has a solid wooden cube with whole number dimensions (in centimeters). He paints the entire surface of the cube blue. Then, with slices parallel to the faces of the cube, Ted cuts it into $1\,\mathrm{cm} \times 1\,\mathrm{cm} \times 1\,\mathrm{cm}$ cubes. A certain number x of the small cubes are completely free of paint. A certain number y of the small cubes are painted on only one side. A certain number z of the small cubes are painted on two sides. Eight cubes have paint on three sides. If $y = 2x$, what was the side length of Ted's original cube?

(A) 3 cm

(B) 4 cm

(C) 5 cm

(D) 8 cm

(E) 10 cm

SOLN $x = (n-2)^3, y = 6 \cdot (n-2)^2$, and $z = 12 \cdot (n-2)$. Solve for n in $6 \cdot (n-2)^2 = 2 \cdot (n-2)^3 \Rightarrow 3 = n-2$. □

19. The altitude a, equal sides b, and non-equal side c of an isosceles triangle have lengths that are, in the order listed, consecutive even numbers of centimeters. What is the area of the triangle?

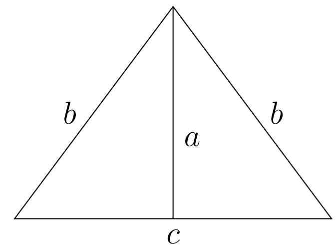
(A) $6\,\mathrm{cm}^2$

(B) $16\,\mathrm{cm}^2$

(C) $30\,\mathrm{cm}^2$

(D) $48\,\mathrm{cm}^2$

(E) $70\,\mathrm{cm}^2$



SOLN Altitude $a = 2n$ for some integer n. The equal sides are $b = 2n+2$, and the base is $c = 2n+4$, so half of it is $n+2$.

The altitude of an isosceles triangle splits it into two right triangles, so the Pythagorean Theorem gives

$$\begin{aligned}
(n+2)^2 + (2n)^2 &= (2n+2)^2 \\
n^2 + 4n + 4 + 4n^2 &= 4n^2 + 8n + 4 \\
n^2 &= 4n
\end{aligned}$$

If $n = 0$, there is no triangle. $n = 4$ gives an area of $\frac{1}{2}(12)(8) = 48\,\mathrm{cm}^2$. □

20. The period T of a pendulum is the time it takes to make one complete oscillation. The period is related to the length L and the acceleration due to gravity $g = 9.8\,\mathrm{m/s^2}$ by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

How long should the pendulum be in order to have a period of $T = \pi$ seconds?

(A) 1 m

(B) 1.5 m

(C) 2.45 m

(D) 9.8 m

(E) 19.6 m

SOLN Solving the period formula gives $L = \left(\frac{T}{2\pi}\right)^2 \cdot g = \left(\frac{\pi\,\mathrm{s}}{2\pi}\right)^2 \cdot 9.8\,\mathrm{m/s^2} = \frac{9.8\,\mathrm{m/s^2}}{4\,\mathrm{s^{-2}}}$ □

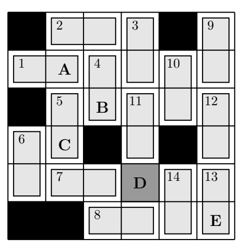
21. Find the value of C such that the parabola $y = x^2 + C$ is tangent to the line $y = x$.

$\boxed{\text{(A)}}$ $\frac{1}{4}$
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) 4

$\boxed{\mathcal{SOLN}}$ $y' = 2x$ has a slope of 1 when $2x = 1$ or $x = 1/2, y = 1/2$. Substitute these values into $y = x^2 + C$ to find C. □

22. In the grid of 1×1 squares, which of the squares A, B, C, D, or E, when blacked out will allow the white squares to be covered by exactly 14 dominoes (1×2 rectangles) with no overlaps or gaps?

(A) A
(B) B
(C) C
$\boxed{\text{(D)}}$ D
(E) E



$\boxed{\mathcal{SOLN}}$ Use trial and error; start left of square A where there must be a domino. □

23. Eric filled $2/3$ of his radiator with antifreeze and then added 4 more quarts (a gallon) of antifreeze. After draining half the antifreeze, he needed 11 quarts of antifreeze to fill the radiator to capacity. How many <u>gallons</u> of antifreeze can the radiator hold?

(A) 4.65
$\boxed{\text{(B)}}$ 4.875
(C) 18.6
(D) 19.565
(E) 78

$\boxed{\mathcal{SOLN}}$ $\frac{1}{2}\left(\frac{2}{3}x + 4\right) + 11 = x \Rightarrow \frac{1}{3}x + 13 = x \Rightarrow 13 = \frac{2}{3}x \Rightarrow x = \frac{39}{2}\text{ qt} = \frac{39}{8}\text{ gal}$ □

24. The complex Pauli spin matrices σ_1, σ_2, and σ_3 appear in quantum mechanics.

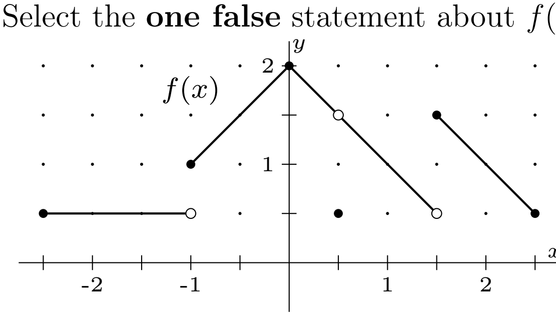
$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Which statement about all three σ_n is **not** true?

(A) $\sigma_n^3 = \sigma_n$
(B) $\det(\sigma_n) = -1$
(C) $\mathrm{Tr}(\sigma_n) = 0$
(D) σ_n is its own multiplicative inverse.
$\boxed{\text{(E)}}$ The product of any two is the third.

$\boxed{\mathcal{SOLN}}$ $\sigma_n^2 = I \implies \sigma_n^3 = \sigma_n$. The product of any two is $\pm\mathrm{i}$ times the third. $\mathrm{Tr}(\sigma_n)$ is the sum of main diagonal elements. □

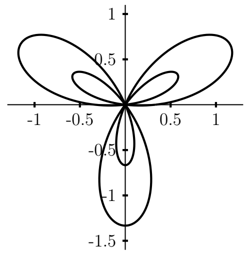
25. Select the **one false** statement about $f(x)$.



(A) $f(-1) = 1$
$\boxed{\text{(B)}}$ $\lim_{x \to -1} f(x) = 1$
(C) $\lim_{x \to 1/2} f(x) = \frac{3}{2}$
(D) $\lim_{x \to 3/2^-} f(x) = \frac{1}{2}$
(E) $\lim_{x \to 3/2^+} f(x) = \frac{3}{2}$

$\boxed{\mathcal{SOLN}}$ $\lim_{x \to -1} f(x)$ does not exist. □

26. Which polar equation best represents the graph for $0 \le \theta \le 2\pi$?



$\boxed{\text{(A)}}$ $r = \sin(3\theta) + \frac{1}{3}$

(B) $r = 6\theta$

(C) $r = \theta^3$

(D) $r = \cos(3\theta) - \frac{1}{3}$

(E) $r = \sin(6\theta)$

SOLN $r = \sin(3\theta)$ gives 6 equal-sized lobes, three of which overlap the other three (and are therefore invisible). Adding the $\frac{1}{3}$ makes the lobes bigger when $\sin 3\theta > 0$ and smaller when $\sin 3\theta < 0$. □

27. What is the output of the following BASIC computer program with some natural number n given as input?

```
10 input n
20 t = 0
30 for i = 1 to n
40 t = t + i
50 next i
60 print t
```

(A) $n!$

(B) the prime factors of n

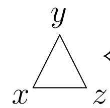
(C) least common multiple of n and f

$\boxed{\text{(D)}}$ nth triangular number 

(E) n^n

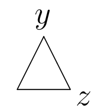
SOLN This is an iterative computation of the nth triangular number T_n.
$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2}$
(If f were initialized to 1 and the operation in line 40 were $*$ instead of $+$ then (A) would be the right answer.) □

28. Definition of the *triangle of power* notation:



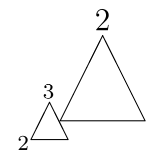
$\Leftrightarrow x^y = z \Leftrightarrow \sqrt[y]{z} = x \Leftrightarrow \log_x z = y$

Any of x, y, z is equivalent to the triangle of power with that number missing, e.g.,



$= x$

Find the value of the following.



(A) $\sqrt{2}$

(B) $\sqrt[3]{8}$

(C) 3

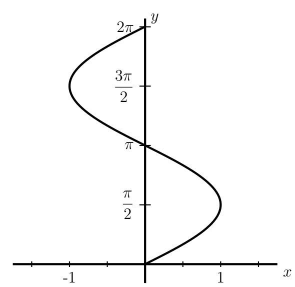
(D) $\sqrt{8}$

$\boxed{\text{(E)}}$ 64

SOLN $2^3 = 8,\ 8^2 = 64$

www.youtube.com/watch?v=sULa9Lc4pck □

29. Which set of parametric equations will produce the graph shown?



(A) $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \end{cases}$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

(B) $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$ $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

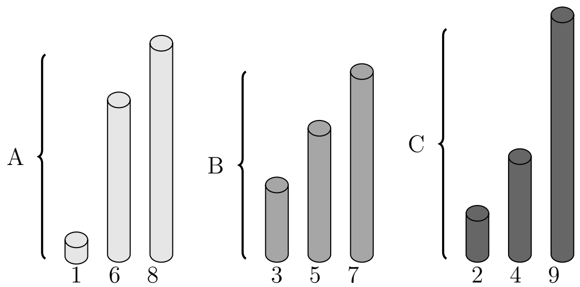
(C) $\begin{cases} x(t) = t \\ y(t) = \cos t \end{cases}$ $\quad -1 \le t \le 1$

(D) $\begin{cases} x(t) = \sin 2t \\ y(t) = \cos t \end{cases}$ $\quad 0 \le t \le 2\pi$

$\boxed{\text{(E)}}$ $\begin{cases} x(t) = \sin t \\ y(t) = t \end{cases}$ $\quad 0 \le t \le 2\pi$

SOLN Plug in beginning, middle, and ending values for t into the equations for x and y to find a few ordered pairs (x, y). □

30. A binary relation $\mathcal{R}$ on a set X is *transitive* if for all $a, b, c \in X$, if $a \mathcal{R} b$ and $b \mathcal{R} c$, then $a \mathcal{R} c$. An example is the $<$ relation on $\mathbb{N}$. Rock, Paper, Scissors is **not** transitive. For individual sticks, "longer than" is transitive; but for **sets** of sticks, the relation "longer than more often" (denoted $\succ$) doesn't need to be. Give the relationships for these sets.



$\boxed{\text{(A)}}$ $C \succ A \succ B \succ C$

(B) $B \succ A \succ C \succ B$

(C) $C \succ B \succ A \succ C$

(D) $B \succ A \succ C \succ B$

(E) $A \succ C \succ B \succ A$

SOLN When comparing any two sets, there are 9 comparisons of individual sticks. In each relation in $C \succ A \succ B \succ C$ the winning set wins 5 of 9. □

31. All numbers in this problem are in base 7. What is $461 + 246$?

(A) 640

(B) 645

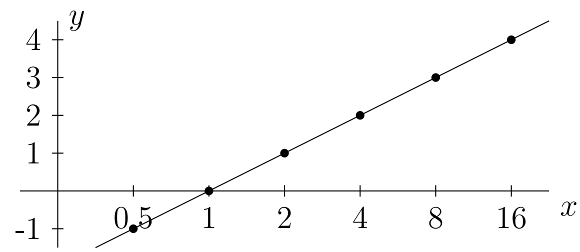
(C) 707

$\boxed{\text{(D)}}$ 1040

(E) 1160

SOLN Start with the 1s place. $1 + 6 = 10$, so carry the 1. Then the 7s place: $1 + 6 + 4 = 14$. Carry the 1. Then the 49s place: $1 + 4 + 2 = 10$. □

32. Which equation best represents the graph?



(A) $y = \sqrt{x}$

(B) $y = 2^x$

$\boxed{\text{(C)}}$ $y = \log_2 x$

(D) $y = 2x + 1$

(E) $y = 2x - 2$

SOLN This is a semilog plot called a lin-log plot because the y-axis is a linear scale and the x-axis is a logarithmic scale. A straight line on a lin-log plot is a logarithmic function. □

33. The sum of three consecutive integers is equal to the product of those integers. How many sets of integers satisfy this condition?

(A) 0

(B) 1

(C) 2

$\boxed{\text{(D)}}$ 3 {-3, -2, -1}, {-1, 0, 1}, {1, 2, 3}

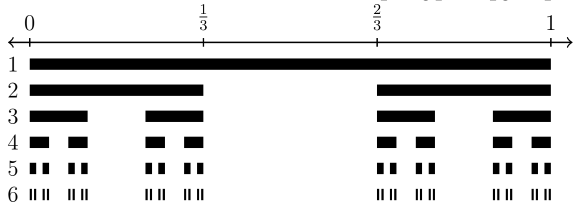
(E) 5

SOLN Let x be the middle number. The three are then $x - 1, x$, and $x + 1$. So

$$\begin{aligned}(x-1)(x)(x+1) &= x - 1 + x + x + 1 \\ x^3 - x &= 3x \\ x^3 - 4x &= 0 \\ x(x^2 - 4) &= 0 \\ x(x-2)(x+2) &= 0 \\ x &= 0, 2, -2\end{aligned}$$

□

34. The Cantor set is constructed by removing in an infinite number of steps the open middle third of each remaining interval of $[0,1]$. For example, the second step is $\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$.



Which is **not** a property of the Cantor set?

(A) It has no intervals.

(B) Between any two numbers in the set, there is a number not in the set.

$\boxed{\text{(C)}}$ It has a finite number of points.

(D) It is self-similar at every scale.

(E) It consists of numbers that, in base 3, contain only 0s and 2s.

SOLN It has an uncountable infinity of points—the same number as $[0,1]$—but each point is isolated, so the set is sparse, rather than dense. Every tiny section of the set at every scale factor looks like the whole set; so its self-similarity makes it very fractal-like; dimension $=\frac{\log 2}{\log 3}$. The points that were removed were the ones that contain 1s in base three.

35. Find the value of the following sum:

$$\sum_{k=1}^{2019}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

(A) $\frac{2016}{2019}$

(B) $\frac{2018}{2019}$

$\boxed{\text{(C)}}$ $\frac{2019}{2020}$

(D) 1

(E) $\frac{2020}{2019}$

SOLN Because of the telescoping nature of the sum, only the last $-\frac{1}{k+1}$ and the first $\frac{1}{k}$ stand yielding

$$\frac{1}{1}-\frac{1}{2020}=\frac{2019}{2020}.$$

36. The following two diagrams are examples of a type of *Young tableaux*.

1	2	3	4	5	6
7					
8					

1	3	5	6	7	8
2					
4					

Diagrams of this type are often used to count complex algebraic symmetries. The rules:

- All integers 1 through 8 must appear in the 8 boxes.
- The numbers in the row must increase to the right
- the numbers in the column must increase downward.

How many possible Young tableaux of this shape are there that follow these rules?

(A) 18

$\boxed{\text{(B)}}$ 21

(C) 24

(D) 32

(E) 35

SOLN Notice that each such diagram is determined completely by the choice of which 2 numbers to have below the corner entry which must always be 1. Therefore the solution is $\binom{7}{2}=21$.

37. Lines are drawn from point E to points B, C, and D in adjacent squares, creating angles α, β, and γ as shown. What is the sum of α, β, and γ?

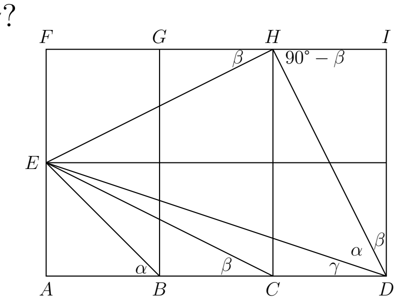
(A) 50°

(B) 60°

(C) 70°

(D) 80°

$\boxed{\text{(E)}}$ 90°



SOLN By symmetry (and that $\triangle EAB$ is right iscoceles) $\alpha=45°$.
Add three new squares on top. $\triangle IDH\cong\triangle ACE$, so $\angle IDH=\beta$.
Examine the angles near H to conclude that $\angle EHD=90° \implies \angle EDH=\alpha$.
$\alpha+\beta+\gamma=\angle ADI=90°$.
www.youtube.com/watch?v=m5evLoL0xwg

38. In the sequence, how many black balls are needed for the 100th term?

(A) 200

(B) 202

(C) 201

(D) 198

(E) 199

SOLN The pattern of the black balls is the number of the term of the sequence times 2; for the 100th term we need 200 black balls. □

39. Which of the following is NOT equivalent to the others?

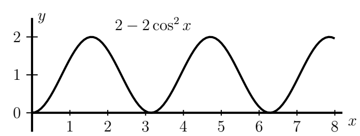
(A) $\tan^2 x$

(B) $\frac{\sin^2 x}{\cos^2 x}$

(C) $\frac{1-\cos 2x}{1+\cos 2x}$

(D) $\sec^2 x - 1$

(E) $2 - 2\cos^2 x$



SOLN The only expression that has a continuous domain is $2 - 2\cos^2 x$. □

40. On a standardized test, Jan scores 70 points. Scores are normally distributed with a mean of 50 and a standard deviation of 20. Which is closest to Jan's percentile?

(A) 20th

(B) 50th

(C) 70th

(D) 85th

(E) 100th

SOLN Jan scored 1 std. dev. above the mean. The empirical rule says 68% of the data are within 1 std. dev. of the mean, so 34% are above the mean and within 1 std. dev. The mean is the 50th %-ile so Jan scored at the 84th %-ile. □