Snow College Mathematics Contest

April 4, 2023 **Senior Division: Grades 10-12** **Form: T**

Bubble in clearly the single best choice for each question you choose to answer.

1. Define $v_3(n) = m$ where $n = 3^m k$, where m and k are positive integers, and 3 is not a factor of k. $v_3(0) = \infty$. Then if $v_3(a) < v_3(b) < v_3(c)$, which is true?

   (A) $v_3(a+b+c) > v_3(a)$

   (B) $v_3(a+b+c) \geq v_3(c) + v_3(a)$

   (C) $v_3(a+b+c) > v_3(c)$

   (D) $v_3(a+b+c) \geq 2v_3(c)$

   (E) $v_3(a+b+c) = v_3(a)$

2. Suppose that a pyramid with a square base is inscribed inside of a sphere of radius 2. What is the area of the base of the pyramid if its base is exactly 1 unit below the center of the sphere? That is, $OE = 1$ and $OS = 2$.

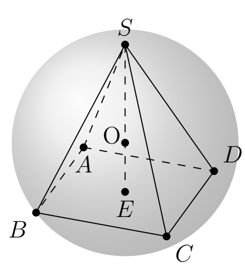
   (A) $3\sqrt{3}$

   (B) $4\sqrt{2}$

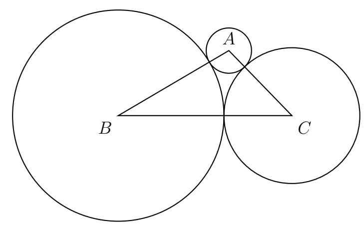
   (C) 6

   (D) $2\sqrt{5}$

   (E) $\sqrt{3} + 4$



3. Three mutually tangent circles have centers A, B, and C and radii a, b, and c respectively. The lengths of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are 17, 23, and 12 respectively. Find the lengths of the radii.



   (A) $a = 13, \quad b = 9, \quad c = 7$

   (B) $a = 5, \quad b = 12, \quad c = 6$

   (C) $a = 6, \quad b = 9, \quad c = 12$

   (D) $a = 4, \quad b = 8, \quad c = 12$

   (E) $a = 3, \quad b = 14, \quad c = 9$

4. A room contains 3 people. Disregarding leap years, what is the probability that at least 2 of the 3 people have the same birthday?

   (A) $\dfrac{365 \cdot 364 \cdot 363}{365^3}$

   (B) $1 - \dfrac{365 \cdot 364 \cdot 363}{365^3}$

   (C) $\dbinom{365}{2}$

   (D) $\dbinom{365}{2} \cdot 2!$

   (E) $\dbinom{365}{2} \cdot 3!$

5. Simplify $\log_{\sqrt{3}} 27$.

   (A) $\sqrt[3]{2}$

   (B) $3/2$

   (C) 3

   (D) 6

   (E) 7

6. Suppose that there are 3 committees. Committees A, B, C have 4, 2, and 1 people respectively. How many ways are there of forming a committee D consisting of exactly 3 members from these committees such that no more than 2 people come from committee A and at least one person comes from committee B?

   (A) 21

   (B) 22

   (C) 23

   (D) 24

   (E) 25

7. Tommy likes to throw darts. He has found that 40% of the time he hits the outer circle (5 points), 30% of the time he hits the middle circle (10 points), 20% of the time he hits the inner circle (20 points), and 5% of the time the bullseye (50 points). 5% of the time he misses the the target (0 points). Each day he throws 3 darts. Which scenario is more likely in a day?

   (A) He hits at least one bullseye

   (B) He hits the bullseye exactly twice

   (C) He scores over 100

   (D) He scores exactly 150

   (E) He misses the target all three times

8. In a class of p students, the average (arithmetic mean) of the test scores is 70. In another class of n students, the average of the scores for the same test is 92. When the scores of the two classes are combined, the average of the test scores is 86. What is the value of $\frac{p}{n}$?

   (A) $\frac{3}{8}$

   (B) 3

   (C) 4

   (D) 8

   (E) Not enough information

9. Find the sum of all the positive factors (divisors) of 400.

   (A) 561

   (B) 759

   (C) 903

   (D) 961

   (E) 981

10. There are several kinds of averages or means. One of them is the geometric mean, which is used often in computing growth rates. The geometric mean of n numbers is defined as $(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{(1/n)}$. Compute the geometric mean of 9, 15, and 25.

    (A) $\frac{49}{3}$

    (B) 15

    (C) $\frac{225}{49}$

    (D) 16

    (E) $\frac{79}{7}$

11. The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of equal size is their element-wise product $A \circ B \equiv [a_{ij}b_{ij}]$. This product differs from the usual matrix product AB. Which of the following is not a true property of the Hadamard product?

(A) $A \circ B = B \circ A$

(B) $A \circ (B + C) = A \circ B + A \circ C$

(C) The Hadamard identity matrix consists of all ones.

(D) The Hadamard inverse exists only if no entry is zero.

(E) $A \circ B = BA$

12. What is the probability that three rolled, fair, six-sided dice will all show different numbers?

(A) $\frac{5}{9}$

(B) $\frac{5}{27}$

(C) $\frac{1}{3}$

(D) $\frac{2}{3}$

(E) $\frac{2}{9}$

13. If $a_{k+2} - a_{k+1} - 2a_k = 0$ for $k = 0, 1, 2, \ldots$, find a_6 if $a_0 = 9$ and $a_1 = -12$.

(A) -54

(B) -32

(C) 27

(D) 54

(E) 64

14. In the following figure, circles A, B, C, and D are all radius 1 and tangent to each other. What is the area of the shaded region?

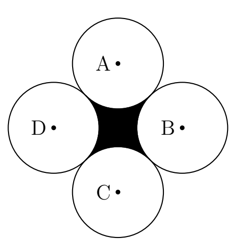
(A) $\frac{\pi}{4}$

(B) 1

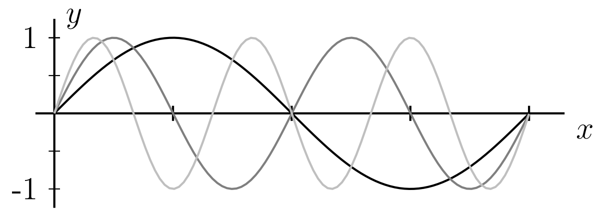
(C) $4 - \pi$

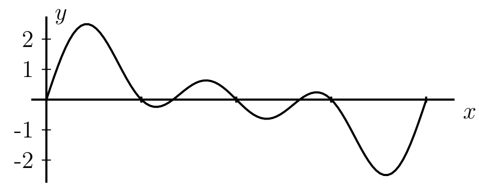
(D) π

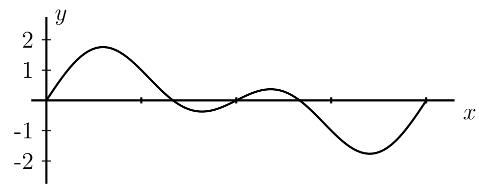
(E) 4

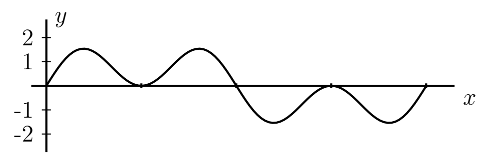


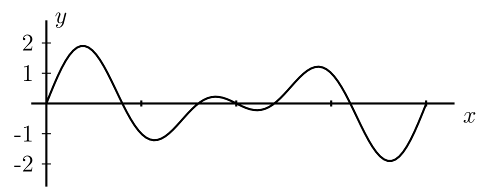
15. Below are graphs of $y = \sin x$, $y = \sin 2x$, and $y = \sin 3x$ on the interval $[0, 2\pi]$. Which of the following graphs shows $y = \sin x + \sin 2x + \sin 3x$?

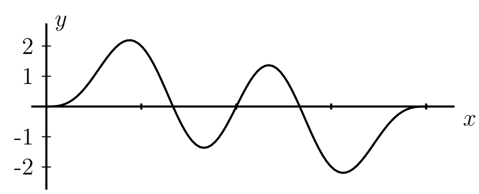


(A) 

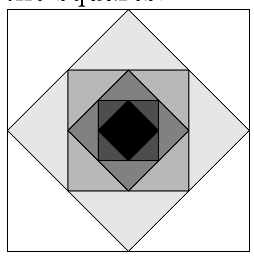
(B) 

(C) 

(D) 

(E) 

16. Only the first six of a sequence of squares are shown. The outermost square has an area of $8\,\text{cm}^2$. Each of the other squares is obtained by joining the midpoints of the sides of the previous square. Find the sum of the infinite series of the areas of all the squares.



(A) $12\,\text{cm}^2$
(B) $14\,\text{cm}^2$
(C) $16\,\text{cm}^2$
(D) $20\,\text{cm}^2$
(E) $100\,\text{cm}^2$

17. Line $y = mx + b$ is tangent to the circle $(x+1)^2+(y-1)^2 = 25$ at $(3, 4)$. Find $m+b$.

(A) $\frac{5}{12}$
(B) $\frac{5}{2}$
(C) $\frac{7}{2}$
(D) $\frac{20}{3}$
(E) $\frac{35}{4}$

18. Average angular velocity is defined as angular displacement divided by time. A year contains approximately $\pi \times 10^7$ seconds. What is the angular velocity of the earth as it travels around the sun?

(A) 10^{-7} rad/s
(B) 2×10^{-7} rad/s
(C) 10^{7} rad/s
(D) 360×10^{-7} rad/s
(E) 180×10^{-7} rad/s

19. Suppose that ☺(n) is a function of positive integers which returns the smallest prime number that is equal to or larger than n. Find ☺(159) + ☺(44).

(A) 203
(B) 210
(C) 213
(D) 222
(E) 238

20. Kim plays basketball for her school. Her free-throw shooting percentage for the season was 76% exactly before today. During tonight's game she makes all five free throws, bringing her percentage up to 80%. How many free throws has Kim made for the entire season (including tonight)?

(A) 24
(B) 25
(C) 26
(D) 28
(E) 30

21. Consider the following recursive function. If $n = 0$, then $f(0) = 1$. For n an integer greater than zero, $f(n) = n \cdot f(n-1)$. What does this recursion compute?

(A) $n!$
(B) n^n
(C) $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$
(D) $n \cdot (n-1) \cdot 1$
(E) n^{n-1}

22. Lucy, Ricky, Ethel, and Fred all go to a party where the following pie options are available: apple, pumpkin, chocolate cream, and pecan. If attendees are allowed to eat at most one piece of each pie flavor, use the following clues to determine which types of pie were eaten by each of the four friends.

- Ricky is allergic to chocolate, and he detests pumpkin pie.
- Ricky and Ethel were the only ones to eat apple pie.
- Lucy ate only one piece of pie while the other three each ate two pieces.
- Lucy and Ethel each ate a piece of chocolate cream pie.
- Only one of the four friends ate a piece of pumpkin pie.
- Between the four friends, no more than two pieces of any one flavor were eaten.
- The four friends ate a combined total of seven pieces of pie.

Which of the friends ate pecan pie?

(A) Ricky, Ethel

(B) Lucy, Fred

(C) Ricky, Lucy

(D) Ricky, Fred

(E) Lucy, Ethel

23. Consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$.

$$\begin{array}{ll} q^1 \div p \text{ has remainder } a_1 \\ q^2 \div p \text{ has remainder } a_2 \\ q^3 \div p \text{ has remainder } a_3 \\ \vdots & \vdots \end{array}$$

No matter the choice of integers p and q, this sequence a_n *always has a repeating pattern*! Use this idea to determine which of the following is true of $2^{241} - 2$.

(A) It must be a multiple of 3 but not 5.

(B) It must be a multiple of 5 but not 3.

(C) It must be a multiple of 3 and 5.

(D) It must be a multiple of neither 3 nor 5.

(E) Not enough information

24. Since 2π shows up in so many physics formulas, it has been proposed that it be replaced with a single symbol τ to simplify.
Evaluate Euler's remarkable formula

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

at $\theta = \tau$ rad.

(A) -1

(B) 0

(C) 1

(D) π

(E) τ

25. An *algebraic number* is any number that is a root of a non-zero polynomial with rational coefficients. The golden ratio $\phi = \frac{1}{2}(1 + \sqrt{5})$ is an irrational algebraic number. Which polynomial is it a root of?

(A) $x^2 - x - 1$

(B) $x + \sqrt{5}$

(C) $x^3 + x^2 + x + 1$

(D) $5x^2 + 2x + 1$

(E) $5x^3 - 2x^2 - x - 1$

26. A *transcendental number* is any number that is **not** a root of a non-zero polynomial with rational coefficients. The imaginary $\sqrt{-1} = \mathrm{i}$ is not transcendental (it is the root of $x^2 + 1$), but i^{i} is. Express i^{i} in an alternate form using Euler's remarkable formula $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and $\tau = 2\pi$.

(A) $\sin^2(\tau) + \cos^2(\tau)$

(B) $\sin(\mathrm{e}^{\tau})$

(C) $\mathrm{e}^{\sin(\tau)}$

(D) $\cos(\tau/2)$

(E) $\mathrm{e}^{-\tau/4}$

27. What is the least number of prime factors (not necessarily different) that 700 must be multiplied by so that the product is a perfect cube?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

28. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find B.

(A) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$

29. What is the area of the largest rectangle that can be inscribed in a closed semicircle of radius 4?

(A) 32

(B) 16

(C) $8\sqrt{2}$

(D) $16\sqrt{2}$

(E) $12\sqrt{3}$

30. If I multiply the number of math contests I have taken in my life by 6 and then add 5, the resulting number cannot be divisible by which number?

(A) 5

(B) 7

(C) 9

(D) 11

(E) 13

31. If you place a cake of soap on a pan of a scale and $\frac{3}{4}$ cake of soap and a $\frac{3}{4}$-kg weight on the other, the pans balance. How much does a cake of soap weigh?

(A) 3 kg

(B) 1 kg

(C) $\frac{3}{4}$ kg

(D) $\frac{1}{2}$ kg

(E) $\frac{1}{4}$ kg

32. Which is the correct description of the graph of $(x + y)^2 = x^2 + xy + y^2$?

(A) Two intersecting lines

(B) The empty set

(C) A single point

(D) Two parallel lines

(E) A circle

33. Which of the following is equivalent to the value of $\tan 10°$?

(A) $\cot 80°$

(B) $\tan 80°$

(C) $\sec 10°$

(D) $\csc 80°$

(E) $\tan 55°$

34. The polar graph shown below is the lemniscate with equation $r^2 = \cos 2\theta$. Which Cartesian equation of conic sections is equivalent to the inverse curve (reciprocal) of the lemniscate?

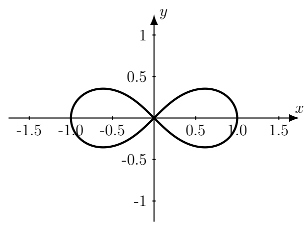
(A) $y = x^2$

(B) $x^2 + y^2 = 1$

(C) $x^2 - y^2 = 1$

(D) $y^2 - x^2 = 1$

(E) $x^2 + y^2/2 = 1$



35. How many distinguishable ways are there to rearrange the letters in the word MINIMUM (example: "MMIINMU").

(A) 70

(B) 120

(C) 240

(D) 420

(E) 840

36. How many solutions are there to $\cos(2x) = \sin(2x)$ on the interval $[0, 2\pi]$?

(A) 0

(B) 1

(C) 2

(D) 4

(E) 6

37. What is the volume of the region formed by revolving the area in the first quadrant between $y = \sqrt{4 - x^2}$ and $y = 2 - x$ around the y-axis?

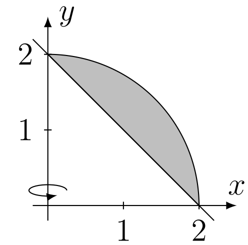
(A) $8\pi/3$

(B) 2π

(C) 3π

(D) $5\pi/2$

(E) $17\pi/6$



38. What is the length of the curve?

$$y = \tfrac{2}{3}(x-1)^{\frac{3}{2}}, \qquad 1 \le x \le 4$$

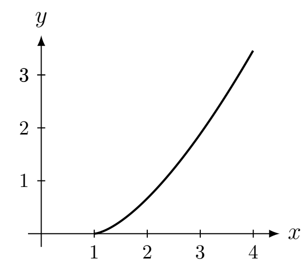
(A) $\frac{12\sqrt{3}}{5}$

(B) 9

(C) $\frac{14}{3}$

(D) 12

(E) $2\sqrt{3}$



39. What is the area between $y = x^2$ and $y = x^3$ between $x = 0$ and $x = 1$?

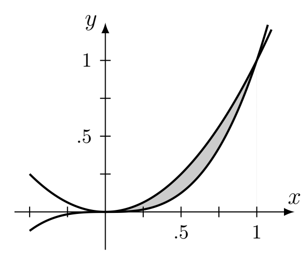
(A) $\frac{1}{12}$

(B) $\frac{\pi}{4}$

(C) $\frac{1}{8}$

(D) $\frac{2}{5}$

(E) $\frac{4}{9}$



40. What is the maximum height that a 20 ft wide truck can have in order to barely pass under a parabolic arch with a height of 25 ft and a base width of 50 ft?

(A) 16 ft

(B) 20 ft

(C) 21 ft

(D) 22 ft

(E) 24 ft